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ON A BIVARIATE GENERALIZED ALTERNATIVE HYPER-POISSON DISTRIBUTION

Muhammad Shoaib[1], **Zafar Iqbal**[2] and **Munir Ahmad**[3]
[1] National University of Modern Languages, Lahore, Pakistan
Email: shoaib_ravian77@yahoo.com
[2] Department of Statistics, Govt. College Satellite Town,
Gujranwala, Pakistan. Email: iqbalzafar825@gmail.com
[3] National College of Business Administration & Economics
Lahore, Pakistan. Email: drmunir@ncbae.edu.pk

ABSTRACT

In this paper we develop a bivariate generalized alternative hyper-Poisson distribution (BGAHPD) and discuss some of its important properties such as factorial moments and certain recurrence relations for its probabilities.

KEY WORDS

Confluent Hyper-geometric Functions, Factorial Moment Generating Function.

1. INTRODUCTION

Bardwell and Crow (1964) obtained a two parameter family of discrete distribution which is named as “Hyper-Poisson distribution”: The probability generating function (pgf) of hyper-Poisson distribution is

$$G(z;\gamma;\theta)=\frac{{}_1F_1(1;\gamma;\theta z)}{{}_1F_1(1;\gamma;\theta)},\ \gamma>0,\theta>0 \tag{1}$$

where ${}_1F_1(a_1;a_2;z)=1+\sum_{k=1}^{\infty}\frac{(a_1)_k z^k}{(a_2)_k k!}$, $(a)_k=a(a+1)...(a+k-1)$ and ${}_1F_1(a_1;a_2;z)$ is the confluent hyper-geometric series (also called Kumar M function).

Staff (1964) discussed the displaced Poisson distribution. Some queuing theory associated with hyper-Poisson distribution arrivals had been worked out by Nisida (1962). Katz (1963) generated family of distributions by including more parameters in the class of discrete distributions. Crow and Bardwell (1965) estimated the parameters of the hyper-Poisson distribution. Kemp (1968) discussed that many discrete distributions may be written in the form of the hyper-geometric series. Ahmad (1992) estimated the parameters of incomplete bivariate hyper-Poisson distribution. Kemp (2002) developed a q-analogue of the hyper-Poisson distribution. Roohi and Ahmad (2003a) estimated the parameters of the hyper-Poisson distribution using negative moments.

Roohi and Ahmad (2003b) derived expressions for ascending factorial moments and further obtained certain recurrence relations for negative moments and ascending factorial moments of the hyper-Poisson distributions. Kumar and Nair (2012) developed an alternative form of hyper-Poisson distribution.

2. SOME WELL KNOWN BIVARIATE DISTRIBUTIONS

Tiecher (1954) gives the recurrence relation for the bivariate Poisson distribution. Holgate (1964) developed a bivariate Poisson distribution with parameters $\lambda_1^* = \lambda_1 + \lambda_3$ *and* $\lambda_2^* = \lambda_2 + \lambda_3$ through the probability generating function.

$$\Pi(t_1, t_2) = \exp\left[\lambda_1^*(t_1 - 1) + \lambda_2^*(t_2 - 1) + \lambda_3(t_1 - 1)(t_2 - 1)\right], \qquad (2)$$

Kocherlakotta and Kocherlakotta (1992, pp.90) discussed bivariate Poisson distribution with parameters $\lambda_1 = (f + h)t$, $\lambda_2 = (g + h)t$, $\lambda_3 = ht$ through the probability generating function.

$$\Pi(t_1, t_2) = \exp\left[(t_1 - 1)(f + h)t + (t_2 - 1)(g + h)t + (t_1 - 1)(t_2 - 1)ht\right], \qquad (3)$$

Ahmad (1981) developed a bivariate version of the hyper-Poisson distribution (which we named as the bivariate hyper-Poisson distribution or in short the BHPD) through the probability generating function (p.g.f)

$$Q(t_1, t_2) = \left[\phi_1(\eta_1)\phi_2(\eta_2)\right]^{-1} \exp\left[\eta(t_1 - 1)(t_2 - 1)\right]\phi_1(\eta_1 t_1)\phi_2(\eta_2 t_2), \qquad (4)$$

where $\phi_i(x) = \phi(1; \lambda_i; x)$. For $r \geq 0$, $s \geq 0$, the p.m.f. $q(r, s) = P\left[Z_1 = r, Z_2 = s\right]$ of $Z = (Z_1, Z_2)$ following BHPD with p.g.f. (3) is

$$q(r,s) = \frac{e^{\eta}\Gamma(\lambda_1)\Gamma(\lambda_2)}{\phi_1(\eta_1)\phi_2(\eta_2)} \sum_{i=0}^{\min(r,s)} \sum_{j=0}^{r-i} \sum_{k=0}^{s-i} \frac{(-1)^{j+k} \eta_1^{r-i-j} \eta_2^{s-i}}{\Gamma(\lambda_1 + r - i - j)} \frac{\eta^{i+j+k}}{\Gamma(\lambda_2 + s - i - k) i!\, j!\, k!},$$

where $\lambda_1 > 0, \lambda_2 > 0$ and $0 < \eta \leq \min\left[\left(\frac{\eta_1}{\lambda_1}\right), \left(\frac{\eta_2}{\lambda_2}\right)\right]$.

Kumar and Nair (2014) discussed the bivariate version of the alternative hyper-Poisson distribution (which we named as the bivariate alternative hyper-Poisson distribution or in short the BAHPD) through the probability generating function (p.g.f)

$$H(t_1, t_2) = \phi\left[1; \gamma; \theta_1(t_1 - 1) + \theta_2(t_2 - 1) + \theta_3(t_1 t_2 - 1)\right] \qquad (5)$$

The p.m.f, $h(r, s)$ following BAHPD with p.g.f. (5) is

$$h(r,s) = \theta_1^r \theta_2^s \sum_{m=0}^{\min(r,s)} \frac{D^*(m+1)\Delta_{r+s-m}(0,0)}{m!(r-m)!(s-m)!} \left(\frac{\theta_3}{\theta_1\theta_2}\right)^m, \qquad (6)$$

where is $D^*(m+1) = \prod_{j=0}^{r+s-m-1} D_j$; and $D_j = \frac{1+j}{\gamma + j}$.

3. GENERALIZED ALTERNATIVE HYPER POISSON DISTRIBUTION

Shoaib et al. (2017) developed a generalized alternative hyper-Poisson (GAHP) distribution. This family of distributions is widely used in the field of accident, contagions phenomena, and biological, birth and death processes, and life testing. The probability generating function (pgf) of generalized alternative hyper-Poisson distribution is

$$G(t) = {}_1F_1\left[\alpha;\gamma;\theta(t-1)\right]$$

and the corresponding probability mass function (pmf) of generalized alternative hyper-Poisson distribution is

$$f(x) = \frac{(\alpha)_x \theta^x}{x!(\gamma)_x} {}_1F_1\left[\alpha+x;\gamma+x;-\theta\right], \quad x = 0,1,2,..., \quad \alpha,\theta,\gamma > 0. \tag{7}$$

When $\alpha = 1$, $f(x)$ reduces to AHP distribution developed by Kumar and Nair (2012). When $\alpha = 1$ *and* $\gamma = 1$ then $f(x)$ becomes a Poisson distribution. When $\gamma = 1$ then $f(x)$ *is* an alternative hyper-Poisson distribution (Shoaib et al. 2013)

$$f(x) = \frac{(\alpha)_x \theta^x}{(x!)^2} {}_1F_1\left[\alpha+x;1+x;-\theta\right], \quad \alpha,\theta,\gamma > 0$$

4. BIVARIATE GENERALIZED ALTERNATIVE HYPER-POISSON (BGAHP) DISTRIBUTION

In this section we develop a bivariate generalized alternative hyper-Poisson distribution (BGAHPD)

$$H(t_1,t_2) = \phi\left[\alpha;\gamma;\beta_1(t_1-1)+\beta_2(t_2-1)+\beta_3(t_1t_2-1)\right]. \tag{8}$$

Following Kumar and Nair (2014) and Kocherlakotta and Kocherlakotta (1992), the p.g.f. of BGAHPD distribution is, let Z be a non-negative integer valued random variable

Define

$$\delta_j = \frac{\beta_j}{\beta}, \textit{ for } j = 1,2,3,...$$

and

$$\beta = \beta_1+\beta_2+\beta_3, \beta_1 > 0, \beta_2 > 0, \quad \beta_3 \geq 0.$$

$$H(t_1,t_2) = G\left\{W(t_1,t_2)\right\}$$

$$H(t_1,t_2) = \phi\left[\alpha;\gamma;\beta_1(t_1-1)+\beta_2(t_2-1)+\beta_3(t_1t_2-1)\right].$$

Case I:

When $\alpha = 1$ in (8), BGAHPD is reduced to bivariate alternative hyper-Poisson distribution.

Case II:

When $\alpha = 1, \gamma = 1$ in (8), BGAHPD is reduced to bivariate Poisson distribution discussed in Kocherlakotta and Kocherlakotta (1992).

5. PROBABILITY MASS FUNCTION OF BGAHP DISTRIBUTION

Differentiating the p.g.f. of BGAHPD $H(t_1, t_2)$ with respect to t_1 and t_2, p times and q times respectively, we get

$$H^{(p,q)}(t_1,t_2) = \left(\prod_{i=0}^{p-1} D_i\right) \sum_{m=0}^{\min(p,q)} \binom{q}{m} \frac{p!}{(p-m)!} \beta_3^m (\beta_1 + \beta_3 t_2)^{p-m}$$

$$\left(\prod_{i=p}^{p+q-m-1} D_i\right) (\beta_2 + \beta_3 t_1)^{q-m} \Delta_{p+q-m}(t_1,t_2) \qquad (9)$$

where

$$D_i = \frac{\alpha + i}{\gamma + i}$$

Putting $(t_1, t_2) = (0,0)$ in (9) and by dividing $p!\, q!$, we get the p.m.f. of the BGAHPD as

$$h(p,q) = \beta_1^p \beta_2^q \sum_{m=0}^{\min(p,q)} \frac{D^*(m+1) \Delta_{p+q-m}(0,0)}{m!(p-m)!(q-m)!} \left(\frac{\beta_3}{\beta_1 \beta_2}\right)^m$$

In which $D^*(m+1) = \prod_{j=0}^{p+q-m-1} D_j \ and \ D_j = \dfrac{\alpha + j}{\gamma + j}$.

Result 5.1: Factorial Moment of BGAHP Distribution

The factorial moment $\mu_{[p,q]}$ of the bivariate generalized alternative hyper-Poisson distribution is obtain by putting $(t_1, t_2) = (1,1)$ in (9)

$$\mu_{[p,q]} = p!q!(\beta_1 + \beta_3)^p (\beta_2 + \beta_3)^q \sum_{m=0}^{\min(p,q)} \frac{D^*(m+1)}{m!(p-m)!(q-m)!} \lambda^m \qquad (10)$$

where $\lambda = \beta_3 (\beta_1 + \beta_3)^{-1} (\beta_2 + \beta_3)^{-1}$.

From (10) we obtain

$$E(Z_1)=\mu_{[1,0]}=\frac{\alpha}{\gamma}(\beta_1+\beta_3)$$

$$E(Z_2)=\mu_{[0,1]}=\frac{\alpha}{\gamma}(\beta_2+\beta_3).$$

6. MARGINAL AND CONDITIONAL PROBABILITY GENERATING FUNCTION OF BGAHP DISTRIBUTION

The marginal probability generating function of Z_1 and Z_2 of the bivariate generalized alternative hyper-Poisson distribution is respectively

$$H_{Z1}(t)=H(t,1)=\phi\left[\alpha;\gamma;(\beta_1+\beta_3)(t-1)\right]$$

and

$$H_{Z2}(t)=H(1,t)=\phi\left[\alpha;\gamma;(\beta_2+\beta_3)(t-1)\right]$$

Let z be a non-negative integer such that $P(Z_1=z)>0.$ On differentiating (8) with respect to t_2, z times and putting $t_1=t$ and $t_2=0,$ we get

$$H^{(0,z)}(t,0)=\left[\beta_2+\beta_3 t\right]^z\left(\prod_{j=0}^{z-1}D_j\right)\Delta_z(t,0)$$

where

$$D_j=\frac{\alpha+j}{\gamma+j}$$

and

$$\Delta_j(t_1,t_2)=\phi\left[\alpha+j;\gamma+j;\beta_1(t_1-1)+\beta_2(t_2-1)+\beta_3(t_1t_2-1)\right]$$

for $j=0,1,2,....$

The conditional probability generating function of Z_1 given $Z_2=z$ is obtain as

$$H_{Z_1|Z_2=z}(t)=\left(\frac{\beta_2+\beta_3 t}{\beta_2+\beta_3}\right)\frac{\phi\left[\alpha+z;\gamma+z;-(\beta_2+\beta_3)+\beta_1(t-1)\right]}{\phi\left[\alpha+z;\gamma+z;-(\beta_2+\beta_3)\right]}$$

$$=H_1(t)H_2(t) \qquad (11)$$

where $H_1(t)$ is the p.g.f. of a binomial random variable with parameters z and $p=\beta_3(\beta_2+\beta_3)^{-1}$ and $H_2(t)$ is the p.g.f. of a random variable following a generalized alternative hyper-Poisson distribution with parameters $\alpha+z,\gamma+z,$ and β_1.

Consequently, from (11) we obtain the following

$$E\left(Z_1 \middle| Z_2 = z\right) = \frac{z\beta_3}{\left(\beta_2 + \beta_3\right)} + \frac{\beta_1 D_z \Delta_{z+1}(1,0)}{\Delta_z(1,0)}$$

$$Var\left(Z_1 \middle| Z_2 = z\right) = \frac{z\beta_2\beta_3}{\left(\beta_2 + \beta_3\right)^2} + \frac{\beta_1 D_z}{\Delta_z^2(1,0)}\left\{D_{z+1}\Delta_z(1,0)\Delta_{z+2}(1,0)\beta_1 + \Delta_z(1,0)\Delta_{z+1}(1,0) - D_z\left[\Delta_{z+1}(1,0)\right]^2\beta_1\right\}$$

The conditional probability generating function of Z_2 given $Z_1 = z$ is obtain as

$$H_{Z_1|Z_2=z}(t) = \left(\frac{\beta_1 + \beta_3 t}{\beta_1 + \beta_3}\right)\frac{\phi\left[\alpha + z; \gamma + z; -\left(\beta_1 + \beta_3\right) + \beta_2(t-1)\right]}{\phi\left[\alpha + z; \gamma + z; -\left(\beta_1 + \beta_3\right)\right]}$$

$$= H_1(t)H_2(t) \qquad (12)$$

where $H_1(t)$ is the p.g.f. of a binomial random variable with parameters z and $p = \beta_3\left(\beta_1 + \beta_3\right)^{-1}$ and $H_2(t)$ is the p.g.f. of a random variable following a generalized alternative hyper-Poisson distribution with parameters $\alpha + z, \gamma + z$ and β_2 .

Consequently, from (12) we obtain the following

$$E\left(Z_1 \middle| Z_2 = z\right) = \frac{z\beta_3}{\left(\beta_1 + \beta_3\right)} + \frac{\beta_2 D_z \Delta_{z+1}(1,0)}{\Delta_z(1,0)}$$

$$Var\left(Z_1 \middle| Z_2 = z\right) = \frac{z\beta_1\beta_3}{\left(\beta_1 + \beta_3\right)^2} + \frac{\beta_2 D_z}{\Delta_z^2(1,0)}\left\{D_{z+1}\Delta_z(1,0)\Delta_{z+2}(1,0)\beta_2 + \Delta_z(1,0)\Delta_{z+1}(1,0) - D_z\left[\Delta_{z+1}(1,0)\right]^2\beta_2\right\}.$$

7. RECURRENCE RELATION FOR BGAHP DISTRIBUTION

The following recurrence relationship holds for Bivariate generalized alternative hyper-Poisson distribution

i) $$H\left(t_1, t_2\right) = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} h\left(p, q, \gamma^*\right) t_1^p t_2^q$$

$$= \phi\left[\alpha; \gamma; \beta_1\left(t_1 - 1\right) + \beta_2\left(t_2 - 1\right) + \beta_3\left(t_1 t_2 - 1\right)\right]$$

where $\gamma^* + j = \left(\alpha + j, \gamma + j\right)$

ii) $$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} h\left(p, q, \gamma^* + 1\right) t_1^p t_2^q$$

$$= \phi\left[\alpha + 1; \gamma + 1; \beta_1\left(t_1 - 1\right) + \beta_2\left(t_2 - 1\right) + \beta_3\left(t_1 t_2 - 1\right)\right]$$

The probability mass function $h\left(p,q,\gamma^*\right)$ of the BGAHPD satisfies the following recurrence relations:

i) $h\left(p+1,0,\gamma^*\right)=\frac{\alpha\beta_1}{\gamma(p+1)}h\left(p,0;\gamma^*+1\right),\ p\ge 0$

ii) $h\left(p+1,q,\gamma^*\right)=\frac{\alpha}{\gamma(p+1)}\left[\beta_1 h\left(p,q;\gamma^*+1\right)\right.$

$\left.+\beta_3 h\left(p,q-1;\gamma^*+1\right)\right],\ p\ge 0, q\ge 1$

iii) $h\left(0,q+1,\gamma^*\right)=\frac{\alpha\beta_2}{\gamma(q+1)}h\left(0,q;\gamma^*+1\right), q\ge 0$

iv) $h\left(p,q+1,\gamma^*\right)=\frac{\alpha}{\gamma(q+1)}\left[\beta_2 h\left(p,q;\gamma^*+1\right)\right.$

$\left.+\beta_3 h\left(p-1,q;\gamma^*+1\right)\right],\ p\ge 1, q\ge 0$.

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