Journal of ISOSS
2017 Vol. 3(1), 45-54

ON THE POWER GENERALIZATION OF LOG-SYMMETRIC SIA DISTRIBUTIONS

Saleha Naghmi Habibullah
Department of Statistics, Kinnaird College for Women, Lahore, Pakistan
Email: salehahabibullah@gmail.com

ABSTRACT

Speaking of distributions of non-negative continuous random variables, some researchers have focused on 'log-symmetric' distributions whereas some others have been interested in distributions 'Self-Inverse at A'. In this paper, we first assert that log-symmetric distributions are essentially SIA and then, applying the power-transformation to the log-symmetric density function $g(y)$, obtain the class of "SIA Log-Symmetric Power distributions". The arbitrariness of the exponent in the power-transformation provides a substantial amount of flexibility in the context of modeling and the preservation of the SIA property provides the capability of developing SIA-estimators that are more efficient than the ordinary 'method of moments-type' estimators.

1. INTRODUCTION

Distributions of non-negative continuous random variables enjoy a prominent position in distribution theory due to the fact that they find applications in important disciplines such as reliability engineering, survival analysis, economics and biology. In general, such distributions are unimodal and positively skewed, and provide useful models for modeling variables such as strength, life-length and income among many others. In this paper, we focus on an interesting class of distributions of non-negative continuous random variables, those for which it is possible to develop estimators of distribution parameters that are more efficient than the currently prevalent estimators. We combine two currently existing nomenclatures 'log-symmetric' and 'SIA' for this class of distributions in order to suggest a new term 'SIA log-symmetric distributions'. We present the idea of applying the power transformation $Y = X^r$, $r \in \mathbb{R}^+$ to SIA log-symmetric random variables X in order to obtain greater flexibility due to the arbitrariness of the exponent and comment on the possibility of the power generalizations of SIA log symmetric distributions surpassing the SIA log-symmetric distributions in the context of obtaining better-fitting models to real data-sets.

2. SOME HISTORY

The origins of the notion of 'SIA log-symmetric distributions' can be traced back to a number of decades. An exploration of the intrinsic nature of non-negative random variables that have exactly the same probability distributions as their reciprocals has been carried out in [1]. A necessary and sufficient condition for the invariance under the reciprocal transformation has been presented, and a structural property of distributions possessing this property has been discussed.

The ξ –normal family of continuous probability distributions which can be regarded as a sub-class of the class of distributions invariant under the reciprocal transformation has been discussed in [2]. Appropriate parameters have been introduced, estimation problems discussed, and a practical application where the property of invariance under the reciprocal transformation is important has been mentioned.

The nomenclature 'Strictly Closed Under Inversion (SCUI)' has been adopted in [3] for this class of distributions. In addition to presenting a set of differential equations for generating such distributions, the authors present characterization theorems for some density functions belonging to this class of distributions, derive a new density belonging to this class, and put forth some properties common to all such distributions.

The nomenclature "self-inverse at *a'* for distributions of Y fulfilling the property $Y/a - a/Y$ has been acquired in [4]. The authors regard the median *a* as the point of reciprocity (or the point of inversion); by putting $a=1,$ they obtain distributions 'self-inverse at unity'.

The abbreviation SIU for distributions 'Self-Inverse at Unity' has been adopted in [5]; The abbreviation 'SIA' for distributions Self-Inverse at A appears in [6] --- replacing the '*a*' given in [4] by 'A'.

Almost contemporaneously to the appearance of the concepts presented in [3], [4], [5], [6], the concept of 'log-symmetry' is presented in [7], defined as $\log Y - \log\theta \sim \log\theta - \log Y$ i.e. $Y/\theta - \theta/Y$ where Y is a positive continuous random variable and θ is the median of the distribution. According to the author, the natural analogue of the 'additive/negative' symmetry on R is a 'multiplicative/reciprocal' symmetry on $\mathbb{R}^+$. The authors goes on to present basic properties of log-symmetry and provides a number of examples of log-symmetric distributions.

From the definition of log-symmetry given in [7], it is obvious that 'log-symmetric' distributions are essentially the ***same*** as the 'self-inverse at *a*' distributions put forth in [4], re-named as 'SIA distributions' in [6]. As such, we propose that the two currently existing nomenclatures 'log-symmetric' and 'SIA' for this class of distributions be ***combined*** in order to adopt the nomenclature '***SIA log-symmetric distributions'***.

3. UTILIZATION OF THE SIU PROPERTY FOR EFFICIENT ESTIMATION OF DISTRIBUTION PARAMETERS

In this section, we point to a number of papers that emerged during the period 2011 to 2014 utilizing the SIU property for developing efficient estimators of distribution parameters. The first paper that has demonstrated the usefulness of the self-inversion property for efficient estimation seems to be [4]. The authors present a simple and convenient way of estimating the cumulative distribution function, assuming that the distribution is self-inverse at unity. The empirical cumulative distribution function proposed by them utilizes not only the sample observations but also their reciprocals, and they assert that the variance of the self-inversion-based ecdf is much smaller than that of the ordinary ecdf.

Efficient estimation of the cumulative hazard function of SIU distributions has been taken up in [8]. The well-known estimators as well as the “SIU estimators” of the cdf and the chf are given in Table 3.1.

Table 3.1

Estimation of ↓	Well-Known Estimator	“SIU Estimator”
CDF	$\hat{F}(y)=\frac{1}{n}\sum_{i=1}^{n}\{[Y_i \le y]\}$	$\breve{F}(y) = \frac{1}{2n}\left[\sum_{i=1}^{n}\{[Y_i \le y]\} + \sum_{j=1}^{n}\left\{\left[{}^{1}/_{Y_j} \le y\right]\right\}\right]$
CHF	$\hat{H}(y)=\sum_{k=1}^{n}\left\{\frac{[Y_{(k)} \le y]}{n-k+1}\right\}$	$\breve{H}(y) = \frac{1}{2}\left[\sum_{k=1}^{n}\left\{\frac{[Y_{(k)} \le y]}{n-k+1}\right\} + \sum_{k=1}^{n}\left\{\frac{[1/Y_{(n+1-k)} \le y]}{n-k+1}\right\}\right]$

Subsequently, a number of papers emerged during the years 2013 and 2014 suggesting modifications to well-known estimators of central tendency, dispersion, skewness and kurtosis. (See [9], [10], [11], [12], [13] and [14].) ‘SIU-estimators’ of measures of central tendency and dispersion presented in [9] and [10] are summarized in Table 3.2.

Table 3.2

Estimation of ↓	Well-Known Estimator	“SIU Estimator”
Central Tendency	Mid-Range= $\frac{Y_0+Y_m}{2}$	Mid-Range $_{\text{SIU}}$ $= \frac{1}{4}\left[(Y_0 + Y_m) + \left({}^{1}/_{Y_0} + {}^{1}/_{Y_m}\right)\right]$
	Mid-Hinge= $\frac{Q_1+Q_3}{2}$	Mid-Hinge $_{\text{SIU}}$ $= \frac{1}{4}\left[(Q_1 + Q_3) + \left({}^{1}/_{Q_1} + {}^{1}/_{Q_3}\right)\right]$
	A.M.= $\frac{\sum_{i=1}^{n} Y_i}{n}$	A.M.$_{\text{SIU}}$ $= \frac{1}{2n}\left[\sum_{i=1}^{n} Y_i + \sum_{j=1}^{n} {}^{1}/_{Y_j}\right]$
Dispersion	$\text{Range} = \text{X}_\text{m} - \text{X}_\text{o}$	$\text{Range}_{\text{SIU}}$ $= \frac{1}{2}\left[(\text{X}_\text{m} - \text{X}_\text{o}) + \left(\frac{1}{\text{X}_\text{o}} - \frac{1}{\text{X}_\text{m}}\right)\right]$
	$\text{IDR} = \text{D}_{90} - \text{D}_{10}$	IDR_{SIU} $= \frac{1}{2}\left[(\text{D}_{90} - \text{D}_{10}) + \left(\frac{1}{\text{D}_{10}} - \frac{1}{\text{D}_{90}}\right)\right]$
	$\text{IQR} = \text{Q}_3 - \text{Q}_1$	IQR_{SIU} $= \frac{1}{2}\left[(\text{Q}_3 - \text{Q}_1) + \left(\frac{1}{\text{Q}_1} - \frac{1}{\text{Q}_3}\right)\right]$

‘SIU-estimators’ of measures of skewness and kurtosis presented in [11], [12], [13] and [14] are summarized in Table 3.3.

Table 3.3

Estimation of ↓	**Well-Known Estimator**	**"SIU Estimator"**
Skewness	$Sk_{Bowley} = \dfrac{Q_1 - 2Q_2 + Q_3}{Q_3 - Q_1}$	$\hat{Sk}^*_{Bowley\_SIU} = \dfrac{\left(Q_3 + Q_3^{-1}\right) - 4 + \left(Q_1 + Q_1^{-1}\right)}{\left(Q_3 - Q_3^{-1}\right) - \left(Q_1 - Q_1^{-1}\right)}$
	$Sk_{Kelley} = \dfrac{P_{90} - 2\tilde{X} + P_{10}}{P_{90} - P_{10}}$	$Sk_{Kelley\_SIU} = \dfrac{\left(P_{90} + P_{90}^{-1}\right) - 4 + \left(P_{10} + P_{10}^{-1}\right)}{\left(P_{90} - P_{90}^{-1}\right) - \left(P_{10} - P_{10}^{-1}\right)}$
Kurtosis	$K = \dfrac{Q_3 - Q_1}{2\left(P_{90} - P_{10}\right)},\ 0 < K < 0.5$	$K_{SIU} = \dfrac{\left[\left(Q_3 - Q_3^{-1}\right) - \left(Q_1 - Q_1^{-1}\right)\right]}{2\left[\left(P_{90} - P_{90}^{-1}\right) - \left(P_{10} - P_{10}^{-1}\right)\right]}$
	$K_{Crow\&Siddiqi} = \dfrac{P_{97.5} - P_{2.5}}{P_{75} - P_{25}}$	$K_{Crow\&Siddiqi\_SIU} = \dfrac{P_{25}P_{75}\left(1 + P_{2.5}P_{97.5}\right)\left(P_{97.5} - P_{2.5}\right)}{P_{2.5}P_{97.5}\left(1 + P_{25}P_{75}\right)\left(P_{75} - P_{25}\right)}$

*The formula contained in Table 3.3 corrects an error in [11].

In all of the papers [4], [8], [9], [10], [11], [12], [13] and [14], Monte Carlo simulation has been carried out in order to show that the sampling distributions of the self-inversion-based estimators are ***narrower*** than the sampling distributions of the corresponding well-known estimators. Results pertaining to the coefficients of variations of the sampling distributions of the Crow and Siddiqui's Coefficient of Kurtosis and the self-inversion-based modified estimator given in [14] are presented in Table 3.4.

As well, in these papers, the self-inversion based estimators and the corresponding well-known estimators have been utilized for fitting SIU distributions to real data, and it has been shown that the self-inversion-based estimators yield a better fit than the corresponding well-known estimators.

4. UTILIZATION OF THE SIA PROPERTY FOR EFFICIENT ESTIMATION OF DISTRIBUTION PARAMETERS

As indicated in [7], every SIA log-symmetric distribution $f(x)$ with median A fulfills the functional equation

$$xf\left(\mathrm{Ax}\right) = \frac{1}{x} f\left(\frac{\mathrm{A}}{x}\right)$$

Also, for every such distribution, each 'lower' quantile X_p is related to the corresponding 'upper' quantile X_{1-p} by the relation $X_p/\mathrm{A} = \mathrm{A}/X_{1\text{-}p}$ where $0 < p \le 0.5$.

Table 3.4

Means, Variances and Coefficients of Variation of the Sampling Distributions of Crow and Siddiqui's Coefficient of Kurtosis and the Modified Estimator when drawing 1000 samples from the Birnbaum Saunders distribution with $\alpha = \beta = 1$

Sample Size n	Sampling Distribution of Crow & Siddiqui's Coefficient of Kurtosis			Sampling Distribution of Self-Inversion-Based Modified Estimator		
	Mean	Variance	C.V.	Mean	Variance	C.V.
50	4.574	2.324	0.333	4.424	1.173	0.245
100	5.495	1.362	0.212	5.021	0.853	0.184
150	4.001	0.561	0.187	4.035	0.334	0.143

Utilizing this property, in the year 2015, a self-inversion-based estimator of the mean of an SIA distribution has been proposed in [6]. The estimator is as follows:

$$\bar{x}_{\text{SIA}} = \frac{\sum_{i=1}^{n} x_i + \hat{A}^2 \sum_{j=1}^{n} x_j^{-1}}{2n}$$

where n is the sample size and $\hat{A}$ is the median of the sample.

With the help of a simulation study based on 1000 samples of size 50 each drawn from the lognormal distribution with $\mu = 2$ and $\sigma = 1$, the authors show that the coefficient of variation of the SIA-estimator of the distribution is a little more than ***one half*** of the coefficient of variation of the ordinary sample mean. The results obtained by them are reproduced in Table 4.1.

Table 4.1

	Sampling Distribution of Ordinary Sample Mean	Sampling Distribution of SIA-Based Modified Mean
Minimum	6.3538	9.4225
Maximum	21.7630	19.3176
Range	15.4092	9.8951
Half-Range	7.7046	4.9476
Mid-Range	14.0584	14.3700
Coefficient of Range	0.5480	0.3443
Mean	12.2390	12.1876
Variance	5.1648	1.6346
Coefficient of Variation	0.1857	0.1049

Utilizing both the SIA estimator and the ordinary, non-SIA estimator, the authors fit the lognormal distribution to a set of data given in [15]. The 29 values considered by them represent the **times to repair** for a construction equipment fleet. The values are reproduced below:

3.20, 11.58, 38.37, 1.83, 6.52, 27.43, 15.93, 1.02, 0.50, 7.25, 1.17, 27.40, 92.90, 62.77, 0.33, 0.33, 0.95, 8.58, 1.00, 0.50, 13.83, 1.72, 10.25, 5.25, 40.02, 31.93, 88.27, 0.50, 4.35.

The histogram of the data is reproduced in Figure 4.1.

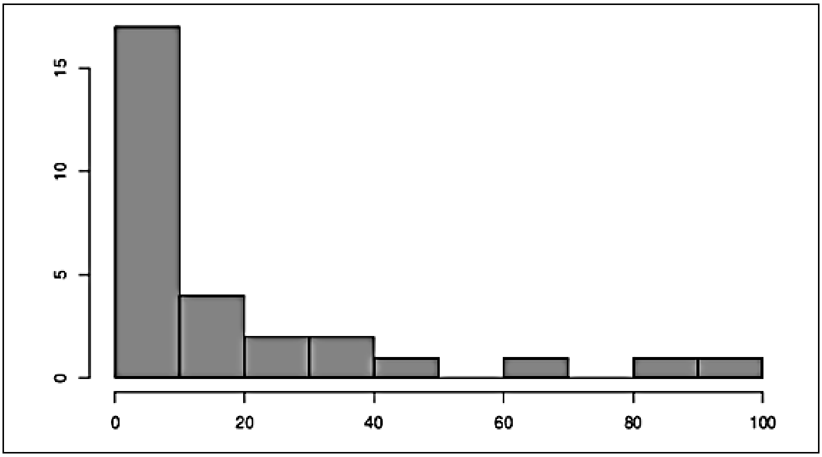

Fig. 4.1: Histogram of the TTR Data

By employing the Kolmogorov Smirnov test in each case, the authors show that the lognormal model based on the SIA estimator fits the data ***better*** than the lognormal model based on the non-SIA estimator.

Subsequently, a number of papers have emerged in 2015 and 2016 in which algebraic expressions of SIA-estimators of parameters of log-logistic, log-Cauchy and lognormal distributions have been proposed and the usefulness of these estimators has been demonstrated through Monte Carlo simulation as well as through fitting to real data (See [16], [17], [18], [19], and [20].)

An SIA-estimator of the Inter-Quartile Mean (IQM), a special type of trimmed mean, one-fourth of the data being discarded on both sides, has been proposed in [21] for the case when a data-set consists of n values such that n is divisible by 4. The author states that, in case of a data-set consisting of n values such that n is divisible by 4, the formula of the Inter-Quartile Mean is given by

$$\text{IQM} = \frac{2}{n} \sum_{i=\frac{n}{4}+1}^{\frac{3n}{4}} x_i \tag{4.1}$$

where the values have been arranged in ascending order. On the other hand, the 'SIA-estimator' is given by

$$\text{IQM}_{\text{SIA}} = \frac{1}{n} \sum_{i=\frac{n}{4}+1}^{\frac{3n}{4}} \left(x_i + \frac{\hat{A}^2}{x_i} \right) \tag{4.2}$$

where the values have been arranged in ascending order, n is divisible by 4 and $\hat{A}$ is the median of the data. By fitting the lognormal distribution to a data set representing the survival times of guinea pigs injected with different doses of tubercle bacilli ([22]), it has been shown in [21] that a better fit is obtained through the use of the SIA-based IQM than the one obtained through the use of the ordinary IQM.

From all of the above, it is obvious that the significance of the newly derived SIA estimators is that, in those situations where the relevant SIA log-symmetric distributions are appropriate for modeling the data at hand, these SIA-estimators are likely to yield ***better-fitting models*** for the data than their well-known counterparts due to the fact that they are ***more efficient*** than their well-known counterparts.

5. SIA LOG-SYMMETRIC POWER DISTRIBUTIONS

In this section, we discuss a class of distributions for which we adopt the nomenclature '***SIA log-symmetric power distributions'***. This class of distributions $f(x)$ is obtained by applying the power-transformation $X = Y^r$ to the log-symmetric density function $g(y)$.

It is easy to show that each distribution $f(x)$ belonging to this class fulfils the functional equation

$$xf\left(\mathrm{A}^r x\right) = \frac{1}{x} f\left(\frac{\mathrm{A}^r}{x}\right) \tag{5.1}$$

where A^r is the median of the distribution. This indicates that the transformed variable X is also self-inverse, the point of inversion (or reciprocity) being A^r.

Remark No. 1: It has been asserted in [7] that if $Y_i, i = 1, 2, ..., m$ are independent random variables each log-symmetric about θ_i, then their product $\prod_{i=1}^{m} Y_i$ is, immediately, log-symmetric about $\prod_{i=1}^{m} \theta_i$. The power transformation referred to above is, obviously, a special case of Jones' result.

Remark No. 2: It has been indicated in [1] that application of the power transformation to a random variable Self-Inverse at Unity preserves the SIU property. It is obvious that the property mentioned in [1] is a special case of the property indicated by Eq. (5.1).

An investigation into the properties of the SIA log-symmetric ***power*** distributions has been commenced. Power generalizations of the 'SIA-lognormal', 'SIA-log-Laplace' and 'SIA-log-Student-t' distributions have been obtained and basic properties have been derived. (See [23], [24] and [25].) Research-work is under way on the power generalizations of the 'SIA-log-logistic' and the 'SIA-log-Cauchy' distributions. In the case of each of these distributions, it is easy to show that the substitution $r = 1$ in the expressions of the means, variances, higher moments and/or quantile functions yields the expressions of the corresponding parameters of the relevant SIA log-symmetric distribution.

6. FUTURE DIRECTIONS

The power transformation referred to above is attractive on two accounts. Firstly, it ***preserves*** the SIA property which provides the capability of developing SIA-estimators that are ***more efficient*** than the ordinary 'method of moments-type' estimators. Secondly, it seems that the SIA log-symmetric power distributions may ***surpass*** the presently prevalent class of SIA log-symmetric distributions in the context of modeling real data due to the ***flexibility*** provided by the exponent which can be an ***arbitrary*** positive real number. As such, future research-work will attempt to determine the extent to which the ***power generalization*** of the class of SIA log-symmetric distributions assists in achieving better-fitting models.

REFERENCES

1. Seshadri, V. (1965). On random variables which have the same distribution as their reciprocals, *Can. Math. Bull.*, 8(6), 819-824.
2. Saunders, S.C. (1974). A family of random variables closed under reciprocation, *J. Amer. Statist. Assoc.*, 69(346), 533-539.
3. Habibullah, S.N., Memon, A.Z. and Ahmad, M. (2010). *On a Class of Distributions Closed under Inversion*, Lambert Academic Publishing (LAP), ISBN 978-3-8383-4868-1.
4. Habibullah, S.N. and Saunders, S.C. (2011). A Role for Self-Inversion, Proceedings of *International Conference on Advanced Modeling and Simulation* (ICAMS, Nov 28-30, 2011) published by Department of Mechanical Engineering, College of Electrical and Mechanical Engineering, National University of Science and Technology (NUST), Islamabad, Pakistan, © 2011, ISBN 978-869-8535-11-7.
5. Habibullah, S.N. (2012). A Generalization of the Standard Half–Cauchy Distribution", Proceedings of the *Twelfth Islamic Countries Conference on Statistical Sciences (ICCS-12)*, Doha, Qatar (Dec 19 to 22, 2012).
6. Habibullah, S.N. and Fatima, S.S. (2015). On a Newly Developed Estimator for More Accurate Modeling with an Application to Civil Engineering, Proceedings of the *12th International Conference on Applications of Statistics and Probability in Civil Engineering (ICASP12)* organized by CERRA (Vancouver, BC, Canada, July 12-15, 2015). Sponsoring Agency: Higher Education Commission, Pakistan.
7. Jones, M.C. (2008). On reciprocal symmetry. *Journal of Statistical Planning and Inference*, 138(10), 3039-3043.
8. Fatima, S.S., Habibullah, S.N. and Saunders, S.C. (2013). Some Results Pertaining to the Hazard Functions of Self-Inverse Life-Distributions; S.N. Habibullah invited to render Oral Presentation of this paper as Keynote Speaker at the *Third International Conference on Aerospace Science and Engineering (ICASE 2013)* (Islamabad, Pakistan, August 21-23, 2013).
9. Fatima, S.S. and Habibullah, S.N. (2013). Self-Inversion-Based Modifications of L-Estimators of Central Tendency for Probability Distributions in the Field of Reliability and Safety, Oral Presentation at the *International Conference on Safety, Construction Engineering and Project Management (ICSCEPM)* (Islamabad, Pakistan, August 19-21, 2013).

10. Fatima, S.S. and Habibullah, S.N. (2013). On Modifications of L-Estimators of Dispersion in the Case of Self-Inverse Distributions, Proceedings of *11th International Conference on Statistical Sciences: Social Accountability, Global Economics and Human Resource Development with Special Reference to Pakistan* (Indus International Institute (NCBA&E Sub-Campus), Dera Ghazi Khan, Pakistan, October 21-23, 2013).
11. Fatima, S.S. and Habibullah, S.N. (2013). On Modifications of the Bowley's Coefficient of Skewness in the Case of Self-Inverse Distributions, Oral Presentation by S.N. Habibullah as Invited Speaker at the *Second National Conference on Space Science organized in order to celebrate the World Space Week* (Karachi, Pakistan, October 7-8, 2013).
12. Habibullah, S.N. and Fatima, S.S. (2014). SIU-Based Modification in Kelley's Measure of Skewness to Achieve Gains in Efficiency, presented at the *Second ISM International Statistical Conference 2014 with Applications in Sciences and Engineering (ISM II)* organized by Faculty of Industrial Sciences and Technology, Universiti Malaysia Pahang, Kuantan, Malaysia on Aug 12-14, 2014; Sponsoring Agency: Higher Education Commission, Pakistan.
13. Habibullah, S.N. and Fatima, S.S. (2014). On Modification of Estimator of the Percentile Coefficient of Kurtosis in the Case of Distributions Self-Inverse at Unity, presented at *12th International Conference on Statistical Sciences: Application of Statistics in Policy Development and Monitoring of Health, Finance, Education, Information Technology and Economics* held at Dow University of Health Sciences, Karachi, Pakistan, March 24-26, 2014.
14. Habibullah, S.N. and Fatima, S.S. (2014). Self-Inversion-Based Modification of Crow and Siddiqui's Coefficient of Kurtosis to Achieve Gains in Efficiency. Proceedings of *Thirteenth Islamic Countries Conference on Statistical Sciences (ICCS-13)* organized by Bogor Agricultural University, Bogor, Indonesia (December 18-21, 2014), Vol. 27, 1-8.
15. Fan, Q. and Fan, H. (2015). Reliability Analysis and Failure Prediction of Construction Equipment with Time Series Models. *Journal of Advanced Management Science*, 3(3), 203-210.
16. Xavier, K. and Habibullah, S.N. (2015). On an SIA-Based Estimator of the Shape Parameter of the Log-Logistic Distribution, Accepted by First International Multi-Disciplinary Conference (IMDC 2015), (Nov. 23-24, 2015, University of Lahore, Gujrat Campus, Gujrat, Pakistan).
17. Ali, A. and Habibullah, S.N. (2015). On a New Estimator of the Scale Parameter of the Log-Cauchy Distribution to Achieve Better Fit, Accepted by *First International Multi-Disciplinary Conference* (IMDC 2015), (Nov 23-24, 2015, University of Lahore, Gujrat Campus, Gujrat, Pakistan).
18. Xavier, K. and Habibullah, S.N. (2016). On A Decile-Based SIA-Estimator of the Shape Parameter of the Log-Logistic Distribution. Proceedings of the *14th International Conference on Statistical Sciences*, Karachi, Pakistan. Vol. 29, 419-426.
19. Ali, A. and Habibullah, S.N. (2016). On A Decile-Based SIA-Estimator of the Scale Parameter of the Log-Cauchy Distribution. Proceedings of the *14th International Conference on Statistical Sciences*, March 14-16, 2016, Karachi, Pakistan. Vol. 29, 427-434.

20. Habibullah, S.N. (2016). On More Accurate Modeling of Corrosion Data of Offshore and Subsea Structures, Proceedings of the *Second International Conference on Safety and Reliability of Ships, Offshore and Subsea Structures* (SAROSS 2016), Aug 15-17, 2016, ASRANet, Glasgow, United Kingdom.
21. Habibullah, S.N. (2015). On a Newly Developed Methodology for More Accurate Modeling. Invited Paper, First International Multi-Disciplinary Conference (IMDC 2015), (Nov. 23-24, 2015, University of Lahore, Gujrat Campus, Gujrat, Pakistan).
22. Bjerkedal, T. (1960). Acquisition of Resistance in Guinea Pies infected with Different Doses of Virulent Tubercle Bacilli. *American Journal of Hygiene*, 72(1), 130-48.
23. Butt, A. and Habibullah, S.N. (2016). On the Derivation of the SIA-log-Student-t distribution for Achieving Better Fit, Proceedings of the *2nd International Multi-Disciplinary Conference* (IMDC 2016), December 19-20, 2016, University of Lahore, Gujrat Campus, Gujrat, Pakistan.
24. Samuel, S. and Habibullah, S.N. (2016). SIA-Lognormal Power Distribution, Proceedings of the *2nd International Multi-Disciplinary Conference* (IMDC 2016), December 19-20, 2016, University of Lahore, Gujrat Campus, Gujrat, Pakistan.
25. Shagufta and Habibullah, S.N. (2016). On the Development of the SIA-Log-Laplace Distribution and its Power Transformation, Proceedings of the *2nd International Multi-Disciplinary Conference* (IMDC 2016), December 19-20, 2016, University of Lahore, Gujrat Campus, Gujrat, Pakistan.